Figure 3 shows the results for ZrO_2 units with grid current collection and copper matrices.

It is clear that T_g/T_c has a marked effect on L/H (hyperbolic); however, it has little effect on δ_3/H within the relevant range of Bi_s and T_g, since this quantity is largely determined by Bi_s.

The numerical data can be approximated with reasonable precision by the following equations:

$$\frac{L}{H}=26.92\mathrm{Bi}_s^{-1.2}\left(\frac{T_g}{T_c}\right)^{-3.13}, \frac{\delta_3}{H}=[-0.0283+0.636\lg \mathrm{Bi}_s]\left(\frac{T_g}{T_c}\right)^{0.1}. \tag{10}$$

Figure 4 shows L/H and Bi_s as functions of T_g/T_c for electrodes with steel matrices; the curves are of falling type, and the following functions provide a close fit for the range T_g/T_c ~1.5-2 for L/H as a function of T_g/T_c and Bi_s as a function of T_g/T_c:

$$\begin{gathered}\frac{L}{H}=7.44\left(\frac{T_g}{T_c}\right)^{-3}+0.418\left(\frac{T_g}{T_c}\right)-0.626,\\ \mathrm{Bi}_s=4.28\left(\frac{T_g}{T_c}\right)^{-2}+0.5\left(\frac{T_g}{T_c}\right)-0.75.\end{gathered} \tag{11}$$

Equations (10) and (11) allow one to define the sizes of the electrode blocks to provide a temperature of $T_c \cong 2000°K$ for a ZrO_2 module for various Bi_s and T_g.

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PROPAGATION OF THERMAL DISTURBANCES IN MEDIA WITH VOLUMETRIC HEAT ABSORPTION

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The solution of one-dimensional unsteady problems of nonlinear heat conduction in the presence of temperature-dependent volumetric heat absorption in the medium is discussed. The conditions are found for the existence of generalized solutions describing temperature waves whose fronts propagate in the medium with a finite velocity.

The investigations carried out in [1, 2] made it possible to formulate the conditions under which the quasilinear equation of heat conduction

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$$\frac{\partial u}{\partial t} = L_n u, \quad L_n u \equiv \frac{\partial^2 u^n}{\partial x^2} \tag{1}$$

has generalized solutions in the form of temperature waves corresponding to the propagation of thermal disturbances with a finite velocity in the medium. The presence of solutions of Eq. (1) in the form of temperature waves presumes, in particular, a null temperature "background" through which the thermal disturbance propagates. In this case for media with $n > 1$ the coefficient of thermal conductivity is reduced to zero at the front of the temperature wave where $u = 0$, which is usually used to explain the finite velocity of propagation of the thermal disturbance [3].

In [4-8] it was indicated that volumetric heat absorption in the medium leads to slowing of the process of propagation of thermal disturbances. It is therefore natural to assume that the finite velocity of propagation of thermal disturbances may be due not only to the reduction to zero of the coefficient of thermal conductivity at the front of the temperature wave, but also to the presence of volumetric heat absorption in the medium. In other words, the conditions under which the velocity of propagation of the fronts of thermal disturbances is finite for processes described by the equation

$$\frac{\partial u}{\partial t} = L_n u - f(u) \tag{2}$$

can differ from the corresponding conditions for Eq. (1). The results presented below confirm this conclusion and show, in particular, that in the presence of volumetric heat absorption a finite velocity of propagation of the fronts of thermal disturbances also occurs for media with $n \leq 1$. In addition, in a number of cases thermal disturbances can also propagate with a finite velocity through a nonzero temperature "background."

We can reduce the study of the unsteady process of propagation of thermal disturbances to the solution of the following boundary-value problem in the region G: $\{x > 0, \ t > 0\}$:

$$\begin{gathered} \frac{\partial u}{\partial t} = L_n u - f(u), \ n > 0, \\ u(x, 0) = U_0 = \text{const} > 0, \\ u(0, t) = \varphi(t), \ u(\infty, t) = U_0. \end{gathered} \tag{3}$$

We will be confined to consideration of the case when $\varphi(t)$ is a nondecreasing function for $t > 0$; $f(u) > 0$ and is continuous for $u > U_0$, and $f(U_0) = 0$. In this case $f(u)$ can have a discontinuity of the first kind at the point $u = U_0$, i.e., in some cases $f(U_0 + 0) \neq 0$. We note that this statement of the problem also presumes that heat fluxes are absent at infinity, i.e.,

$$\left.\frac{\partial u^n}{\partial x}\right|_{x \to \infty} = 0. \tag{4}$$

We can solve the boundary-value problem (3) using the Rothe method of straight lines, which is absolutely stable and has an error $O(\tau)$ [9], where τ is the discrete step in the time variable. Convergence of the sequence of functions constructed by the Rothe method to the solution of the boundary-value problem (3) can be brought about by using procedures analogous to those used in [10].

We introduce a grid of straight lines $t = t_k = k\tau (k = 0, 1, \ldots,)$ with some sufficiently small step $\tau > 0$. Then, after the time variable t is made discrete, we obtain a system of ordinary differential equations for the determination of the approximate values $u_k(x)$ of the function $u(x, t)$ at the points $t = t_k$:

$$\begin{gathered} L_n u_k = \frac{u_k - u_{k-1}}{\tau} + f(u_k), \ k = 1, 2, \ldots, \\ u_0 = U_0 \end{gathered} \tag{5}$$

with the boundary conditions

$$u_k(0) = \varphi(t_k), \ u_k(\infty) = U_0. \tag{6}$$

Making the substitution $v_k = u_k^n$, we write (5) and (6) in the form

$$\frac{d^2 v_k}{dx^2} = \frac{v_k^{\frac{1}{n}} - v_{k-1}^{\frac{1}{n}}}{\tau} + F(v_k),\ k = 1,\ 2,\ \ldots,$$
$$v_0 = V_0,\ v_k(0) = \Phi(t_k),\ v_k(\infty) = V_0, \tag{7}$$

where

$$F(v_k) \equiv f(v_k^{\frac{1}{n}}),\ \Phi(t_k) \equiv [\varphi(t_k)]^n,\ V_0 \equiv U_0^n.$$

In solving the boundary-value problems (7) for k = 1, 2, ..., one successively determines the functions $v_1(x)$, $v_2(x)$, etc.

One can show that

$$v_k(x) \geqslant v_{k-1}(x). \tag{8}$$

Let us prove (8) for k = 1. Assuming the opposite, we find that the function $z = v_1(x) - V_0$ must reach a negative minimum at some point $x = x^0$, and at $x = x^0$

$$z < 0,\ \frac{dz}{dx} = 0,\ \frac{d^2 z}{dx^2} \geqslant 0.$$

On the other hand, using (7) we have

$$\frac{d^2 z}{dx^2} = F(v_1) + \frac{v_1^{\frac{1}{n}} - V_0^{\frac{1}{n}}}{\tau} \tag{9}$$

and one can always take τ small enough that the right side of (9) is negative at $x = x^0$. The contradiction obtained proves that $z \geq 0$ at any point x. Using this result one can similarly prove (8) for all $k > 1$.

It is obvious that the velocity of propagation of the front of a thermal disturbance will be finite for the process described by (3) if for any k one can find $x_k < \infty$ such that $v_k(x) \equiv V_0$ for $x \geq x_k$, where

$$v_k(x_k - 0) = V_0,\ \left.\frac{dv_k}{dx}\right|_{x \to x_k - 0} = 0 \tag{10}$$

at the point $x = x_k$ of the temperature wave front. [The condition (10) assumes that the requirements of continuity of the temperature and heat flux at the temperature wave front are satisfied.]

The first step (k = 1) is decisive in the determination of the conditions under which the velocity of propagation of the fronts of thermal disturbances is finite, since in the case of an infinite velocity the point $x = x_1 < \infty$ of the front, which separates the half-space $x > 0$ into a region $x < x_1$ which thermal disturbances from the wall $x = 0$ have reached in a time τ and an undisturbed region $x \geq x_1$ where $v_1 \equiv V_0$, would already be absent in the first step. At k = 1 we have from (7)

$$\frac{d^2 v_1}{dx^2} = \frac{v_1^{\frac{1}{n}} - V_0^{\frac{1}{n}}}{\tau} + F(v_1),$$
$$v_1(0) = \Phi(\tau),\ v_1(\infty) = V_0. \tag{11}$$

Since $v_1 \equiv V_0$ satisfies Eq. (11), the boundary-value problem (11) can have a front solution only in the case when the solution $v_1 \equiv V_0$ is a singular solution of this equation [7, 8]. In this case the generalized solution $v_1(x)$ of the problem (11), corresponding to a temperature wave with a finite velocity of propagation of the front, will represent a particular solution of Eq. (11) for $x < x_1$, satisfying the boundary condition at $x = 0$ and the conditions (10), joined at the point $x = x_1$ of the front to the singular solution $v_1(x) \equiv V_0$ for $x \geq x_1$.

The generalized solution of the problem (11) can be written in the form

$$\int_{V_0}^{v_1} \frac{ds}{\sqrt{2R(s)}} = x_1 - x \text{ for } x < x_1,$$
$$v_1 \equiv V_0 \text{ for } x \geqslant x_1, \tag{12}$$

where

$$R(s) = \frac{n}{n+1} \cdot \frac{s^{\frac{n+1}{n}} - V_0^{\frac{n+1}{n}}}{\tau} - \frac{V_0^{\frac{1}{n}}}{\tau}(s - V_0) + \int_{V_0}^{s} F(v)\,dv,$$

$$x_1 = \int_{V_0}^{\Phi(\tau)} \frac{ds}{\sqrt{2R(s)}}. \tag{13}$$

The point $s = V_0$ is the only singular point of the integrands in (12) and (14), since with allowance for the limitations imposed on the functions φ and f we have

$$R(s) > 0 \quad \text{for} \quad V_0 < s \leqslant \Phi(\tau). \tag{14}$$

Thus, in order for the solution of the problem (11) to have the form of a front solution characteristic of a temperature wave with a finite velocity of propagation of the front, it is necessary that the improper integral in (14) have an integrable singularity at the point $s = V_0$. Therefore, a front solution of the problem (11) exists if as $s \to V_0$

$$R(s) = O[(s - V_0)^{\alpha}], \tag{15}$$

where $\alpha < 2$. The condition (15) imposes corresponding limitations on V_0 and f under which the solution of the problem (11) will be a front solution.

First of all, when $f(u) \equiv 0$ the condition (15) is satisfied if $V_0 = 0$ ($U_0 = 0$) and $n > 1$. Thus, in the absence of volumetric heat absorption in the medium temperature waves with a finite velocity of propagation of the front can propagate only through a null "background" due to the reduction to zero of the coefficient of thermal conductivity at the wave front.

With $V_0 = 0$ and $n > 1$ front solutions exist for any $f(u)$, where it follows from (14) that volumetric heat absorption in these cases always decreases the velocity of propagation of the temperature wave front. If $n \leq 1$ or $V_0 > 0$ then the coefficient of thermal conductivity is not reduced to zero at the temperature wave front. In these cases the problem (11) can also have front solutions when volumetric heat absorption occurs in the medium ($f \not\equiv 0$). In particular, with $V_0 = 0$ ($U_0 = 0$) and $n \leq 1$ a front solution exists if $f(u) = 0(u^{\beta})$, where $\beta < n$, as $u \to 0$. The case of $\beta \to 0$ corresponds to a discontinuity of the first kind in the function f at the point $u = 0$, and since $n > 0$ in the cases under consideration, in the presence of a discontinuity in f a finite velocity of propagation of thermal disturbances will occur for any n.

If the condition (15) is satisfied, then the expression (12) determines some function $v_1(x)$ which is continuous together with its first derivative for any $x > 0$. The second derivative of the function $v_1(x)$ is continuous everywhere except perhaps for the point $x = x_1$, where it loses continuity if the function f has a discontinuity at the point $u = U_0$.

An analysis of the solutions of the problem (7) for $k > 1$ analogous to that carried out for $k = 1$ shows that when the conditions found above are satisfied the generalized solutions of the problem (7) will also be front solutions in the subsequent steps, where the front of the temperature wave will propagate with a finite velocity through the undisturbed background ($x_k \leq x_{k+1}$).

As an illustration of the above method of solving unsteady problems of nonlinear heat conduction of the type of (3) let us consider in the region $G: \{x > 0,\ t > 0\}$ the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u\theta(u - U_0), \tag{16}$$

$$u(x, 0) = U_0, \quad u(0, t) = U_*, \quad u(\infty, t) = U_0,$$

where U_0 and U_* are certain constants with $U_* > U_0$ and $\theta(z)$ is a step function which for $z \geq 0$ is determined as

$$\theta(z) = \lim_{\gamma \to 0} z^{\gamma} = \begin{cases} 1 & \text{for} \quad z > 0, \\ 0 & \text{for} \quad z = 0. \end{cases}$$

Applying the Rothe method of straight lines, we have the following system of boundary problems for $u_k = u(x, k\tau)$:

$$u_0 = U_0,$$
$$\frac{d^2u_k}{dx^2} = u_k\theta(u_k - U_0) + \frac{u_k - u_{k-1}}{\tau}, \quad (17)$$
$$u_k(0) = U_*, \; u_k(\infty) = U_0, \; k = 1, 2, \ldots$$

For any k the generalized solution of the problem (17) can be obtained in the analytical form

$$u_k(x) = \begin{cases} P_{k-1}^{(k)}(x)\,e^{\lambda x} + Q_{k-1}^{(k)}(x)\,e^{-\lambda x} + \rho_1^{(k)}, & 0 \leqslant x \leqslant x_1, \\ P_{k-2}^{(k)}(x)\,e^{\lambda x} + Q_{k-2}^{(k)}(x)\,e^{-\lambda x} + \rho_2^{(k)}, & x_1 \leqslant x \leqslant x_2, \\ \cdots & \\ P_0^{(k)}(x)\,e^{\lambda x} + Q_0^{(k)}(x)\,e^{-\lambda x} + \rho_k^{(k)}, & x_{k-1} \leqslant x \leqslant x_k, \\ U_0, & x \geqslant x_k, \end{cases} \quad (18)$$

where the constants $\rho_j^{(k)}$ and x_k and the coefficients of the polynomials

$$P_j^{(k)}(x) = a_{j,j}^{(k)}x^j + a_{j,j-1}^{(k)}x^{j-1} + \ldots + a_{j,1}^{(k)}x + a_{j,0}^{(k)},$$
$$Q_j^{(k)}(x) = b_{j,j}^{(k)}x^j + b_{j,j-1}^{(k)}x^{j-1} + \ldots + b_{j,1}^{(k)}x + b_{j,0}^{(k)}$$

are determined from the recurrent equations

$$x_0 = 0, \; \lambda = \left[\frac{1+\tau}{\tau}\right]^{\frac{1}{2}}, \; \rho_k^{(k-1)} = U_0, \; \rho_j^{(k)} = \frac{\rho_j^{(k-1)}}{1+\tau}, \; j = 1, 2, \ldots, k,$$
$$a_{j,j}^{(k)} = -\frac{a_{j-1,j-1}^{k-1}}{2j\lambda\tau}, \; a_{j,\nu}^{(k)} = -\frac{[\nu(\nu+1)\tau a_{j,\nu+1}^{(k)} + a_{j-1,\nu-1}^{(k-1)}]}{2\nu\lambda\tau},$$
$$b_{j,j}^{(k)} = \frac{b_{j-1,j-1}^{k-1}}{2j\lambda\tau}, \; b_{j,\nu}^{(k)} = \frac{[\nu(\nu+1)\tau b_{j,\nu+1}^{(k)} + b_{j-1,\nu-1}^{(k-1)}]}{2\nu\lambda\tau},$$
$$\nu = j-1, \; j-2, \ldots, 2, 1, \quad j = k-1, \; k-2, \ldots, 2, 1,$$
$$\varphi_\nu(x) = [a_{k-\nu,\,k-\nu}^{(k)}x^{k-\nu} + a_{k-\nu,\,k-\nu-1}^{(k)}x^{k-\nu-1} + \ldots + a_{k-\nu,1}^{(k)}x]\,e^{\lambda x}$$
$$+ [b_{k-\nu,\,k-\nu}^{(k)}x^{k-\nu} + b_{k-\nu,\,k-\nu-1}^{(k)}x^{k-\nu-1} + \ldots + b_{k-\nu,\,1}^{(k)}x]\,e^{-\lambda x} + \rho_\nu^{(k)}, \; \nu = 1, 2, \ldots, k-1, \; \varphi_k(x) = \rho_k^k;$$
$$A_1^{(1)} = 0, \; A_{j+1}^{(k)} = \frac{1}{2}\sum_{\nu=1}^{j}\left\{\varphi_\nu(x_\nu) - \varphi_{\nu+1}(x_\nu) + \frac{[\varphi_\nu'(x_\nu) - \varphi_{\nu+1}'(x_\nu)]}{\lambda}\right\}e^{-\lambda x_\nu},$$
$$B_1^{(1)} = 0, \; B_{j+1}^{(k)} = \frac{1}{2}\sum_{\nu=1}^{j}\left\{\varphi_\nu(x_\nu) - \varphi_{\nu+1}(x_\nu) - \frac{[\varphi_\nu'(x_\nu) - \varphi_{\nu+1}'(x_\nu)]}{\lambda}\right\}e^{\lambda x_\nu},$$
$$j = 1, 2, \ldots, k-1,$$
$$C^{(k)} = A_k^{(k)} + \left\{[A_k^{(k)} + B_k^{(k)} + U_* - \rho_1^{(k)}]^2 - \frac{U_0^2}{\lambda^4}\right\}^{\frac{1}{2}},$$
$$a_{k-1,0}^{(k)} = \frac{1}{2}(U_* - \rho_1^{(k)} + B_k^{(k)} - C^{(k)}), \; b_{k-1,0}^{(k)} = \frac{1}{2}(U_* - \rho_1^{(k)} - B_k^{(k)} + C^{(k)}),$$
$$a_{k-j,0}^{(k)} = a_{k-1,0}^{(k)} + A_j^{(k)}, \; b_{k-j,0}^{(k)} = b_{k-1,0}^{(k)} + B_j^{(k)}, \; j = 2, 3, \ldots, k,$$
$$x_k = \frac{1}{\lambda}\ln\left\{\frac{\lambda^2[U_* - \rho_1^{(k)} + B_k^{(k)} + C^{(k)}]}{U_0}\right\}. \quad (19)$$

It is obvious that the solutions (18) have the form of front solutions. It follows from (19), in particular, that for $U_0 \neq 0$ the velocity of propagation of thermal disturbances in the problem (16) is finite.

The distributions $u_k(x)$ for different values of k with $U_0 = 1$, $U_* = 2$, and $\tau = 10^{-3}$ are presented in Fig. 1. These distributions correspond to a temperature wave whose front $x = X_0(t)$ propagates through the nonzero undisturbed temperature "background" with a finite velocity (the position of the front at different times is marked with arrows in the figure). In this case the coefficient of thermal conductivity is a constant which does not depend on the temperature and is not reduced to zero at the temperature wave front. The nature of the motion of the temperature wave front is represented in Fig. 2.

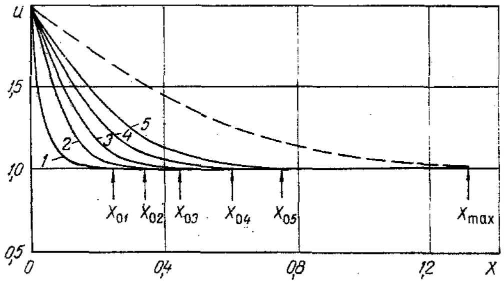

Fig. 1

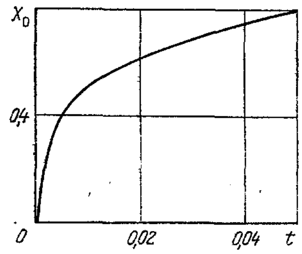

Fig. 2

Fig. 1. Temperature distribution at different times: 1) $t_1 = 0.001$; 2) $t_2 = 0.005$; 3) $t_3 = 0.01$; 4) $t_4 = 0.02$; 5) $t_5 = 0.04$.

Fig. 2. Motion of temperature wave front.

We also note that the evolutionary problem (16) has as a limit as $t \to \infty$ the steady solution [4]

$$u_+(x) = \begin{cases} U_0 \operatorname{ch}(x_{\max} - x) & \text{for } x < x_{\max}, \\ U_0 & \text{for } x \geqslant x_{\max}, \end{cases}$$

where the quantity $x_{max} = \operatorname{arccosh} U_*/U_0$ determines the maximum depth of penetration of the thermal disturbances from the wall. The steady distribution $u_+(x)$ is shown with a dashed line in Fig. 1.

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